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Reduction of the unitary group to its orthogonal or symplectic subgroup: a unified approach based upon complementary groups

C Quesne†

Physique Théorique et Mathématique CP 229, Université Libre de Bruxelles, Bd du Triomphe, B1050 Brussels, Belgium

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Abstract. A unified analysis of the state labelling problems for the d-row irreducible representations of $U(n)$, when reduced with respect to either its orthogonal subgroup $O(n)$ or its symplectic subgroup $USp(n)$ (the latter in the even-n case), is carried out by using appropriate metrics and the complementarity relationship between the groups $O(n)$ and $Sp(2d, R)$, or $USp(n)$ and $SO^*(2d)$. In this way, a recently proposed canonical solution to the $U(n) \supset O(n)$ state labelling problem is extended to the $U(n) \supset USp(n)$ chain. This shows the equivalence between both these internal state labelling problems and the external state labelling problem for $U(d)$, as expressed in Littlewood's branching rules for $U(n) \supset O(n)$ and $U(n) \supset USp(n)$.

1. Introduction

The construction of bases for the d-row irreducible representations (irreps) of the unitary group $U(n)$ is difficult when $U(n)$ is reduced to its orthogonal or symplectic subgroup, because the latter does not provide enough quantum numbers to completely specify the states: this is the so-called state labelling problem.

In a series of papers (Deenen and Quesne 1983, Quesne 1984), a new solution to the $U(n) \supset O(n)$ state labelling problem was proposed. This solution, based upon the complementarity relationship between the $U(n) \supset O(n)$ and $Sp(2d, R) \supset U(d)$ chains (Moshinsky and Quesne 1971, Gross and Kunze 1977, Kashiwara and Vergne 1978), was termed canonical because it reflects in a very simple way the reduction of the internal state labelling problem for $U(n) \supset O(n)$ to the external state labelling problem for $U(d)$, as expressed in Littlewood's branching rule (1950), supplemented, when necessary, with Newell's modification rules (1951). It is not restricted to small values of n or d, although its practical usefulness is limited by the need for an explicit knowledge of some $U(d)$ coupling and recoupling coefficients.

The aim of the present paper is to extend such a solution to the case of $U(n) \supset USp(n)$, where n is even. For such a purpose, we shall realise the generators of the $U(n)$ orthogonal and symplectic subgroups in terms of boson creation and annihilation operators, defining them in a unified way by using appropriate metrics (Quesne 1985). The previously obtained results for $O(n)$ will then easily be transposed for $USp(n)$,

† Maître de recherches FNRS.

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provided we replace the $Sp(2d, R) \supset U(d)$ chain by the $SO^*(2d) \supset U(d)$ one, complementary with respect to $U(n) \supset USp(n)$ (Gelbart 1979, Quesne 1985).

In § 2, the unified treatment of the unitary group orthogonal and symplectic subgroups, as well as their complementary groups, is reviewed. The reduction of $U(n)$ to $O(n)$ or $USp(n)$ is then analysed in § 3. In § 4, bases of the $O(n)$ or $USp(n)$ scalar irreps, belonging to the carrier space of a $U(n)$ d-row irrep, are constructed. They are finally used in § 5 to solve the state labelling problem for $U(n) \supset O(n)$ or $USp(n)$, where arbitrary irreps of the subgroup are considered.

2. The orthogonal and symplectic subgroups of U(n) and their complementary groups

In a previous paper (Quesne 1985), the orthogonal and sympletic subgroups of $U(n)$ were treated in a unified way by introducing a metric $g = \|g_{st}\|$, satisfying the conditions $g\tilde{g} = I$ and $\tilde{g} = \varepsilon g$, where $\sim$ stands for transposed, and $\varepsilon = +1$ for $O(n)$ and -1 for $USp(n)$. For the present purpose, it is convenient to make a definite choice for g. Let us denote the n values of the indices s, t by $\nu, \nu-1, \ldots, 1, -1, \ldots, -\nu+1, -\nu$ when $n = 2\nu$, and $n = \nu, \nu-1, \ldots, 1, 0, -1, \ldots, -\nu+1, -\nu$ when $n = 2\nu+1$, and let us choose

$$\begin{aligned} g &= \bar{I}_n && \text{for } O(n), n = 2\nu \text{ or } 2\nu+1 \\ &= \begin{pmatrix} O_\nu & \bar{I}_\nu \\ -\bar{I}_\nu & O_\nu \end{pmatrix} && \text{for } USp(n),\ n = 2\nu. \end{aligned} \tag{2.1}$$

Here $\bar{I}_k$ is the $k \times k$ matrix with $+1$ on the minor diagonal, and 0 elsewhere. We can therefore write

$$g_{st} = \sigma_s \delta_{s,-t} \tag{2.2}$$

where σ_s is equal to 1 for $O(n)$, and to the sign of s for $USp(n)$.

To realise the $U(n)$ d-row irreps, we need at least dn boson creation and annihilation operators (Baird and Biedenharn 1963, Moshinsky 1963). Let us denote them by η_{is} and ξ_{is}, $i = 1, \ldots, d$, $s = 1, \ldots, n$, respectively. In terms of the $U(n)$ generators

$$C_{st} = \sum_{i=1}^{d} \eta_{is}\xi_{it} \tag{2.3}$$

those of $O(n)$ and $USp(n)$ are defined by

$$\Lambda_{st} = \sigma_s C_{-s,t} - \varepsilon\sigma_t C_{-t,s} = -\varepsilon\Lambda_{ts} = \varepsilon\sigma_s\sigma_t(\Lambda_{-t,-s})^\dagger \tag{2.4}$$

and their commutation relations are given by

$$[\Lambda_{st}, \Lambda_{s't'}] = \sigma_{s'}\delta_{s',-s}\Lambda_{t't} + \sigma_{t'}\delta_{t',-t}\Lambda_{s's} + \sigma_{s'}\delta_{s',-t}\Lambda_{st'} + \sigma_{t'}\delta_{t',-s}\Lambda_{ts'}. \tag{2.5}$$

We can take as independent generators the operators Λ_{st}, where $s > t$ for $O(n)$ and $s \geqslant t$ for $USp(n)$. They separate into the following three subsets:

(i) $\Lambda_{s,-s}$, $\quad s = \nu, \ldots, 1,$ (2.6a)

(ii) Λ_{st}, $\quad s > t > -s$ for $O(n)$, and $s \geqslant t > -s$ for $USp(n)$ (2.6b)

(iii) $>_{st}$, $\quad -t > s > t$ for $O(n)$, and $-t > s \geqslant t$ for $USp(n)$

respectively made of the weight, raising and lowering generators.

To construct bases for the d-row irreps of $U(n)$, reduced to $O(n)$ or $USp(n)$, it is convenient to consider the complementary groups of $U(n)$, and $O(n)$ or $USp(n)$, contained in the group $Sp(2dn, R)$ generated by all bilinear operators in η_{is} and ξ_{is} (Moshinsky and Quesne 1970, Howe 1979). For $U(n)$, it is the group $U(d)$ (Moshinsky 1963), whose generators† are defined by

$$C_{ij} = \sum_{s=-\nu}^{\nu} \eta_{is}\xi_{js} \tag{2.7}$$

and are the weight, raising and lowering generators according to $i=j$, $i<j$, and $i>j$. For $O(n)$ and $USp(n)$, the complementary groups are respectively the groups $Sp(2d, R)$ and $SO^*(2d)$ (Moshinsky and Quesne 1971, Gross and Kunze 1977, Kashiwara and Vergne 1978, Gelbart 1979, Quesne 1985), both generated by the operators

$$\begin{aligned} D^\dagger_{ij} &= \sum_s \sigma_s \eta_{is}\eta_{j,-s} \\ D_{ij} &= \sum_s \sigma_s \xi_{is}\xi_{j,-s} \end{aligned} \tag{2.8}$$

and

$$E_{ij} = \tfrac{1}{2}\sum_s (\eta_{is}\xi_{js} + \xi_{js}\eta_{is}) = C_{ij} + \tfrac{1}{2}n\delta_{ij}.$$

The latter satisfy the following commutation relations:

$$\begin{aligned} &[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj} \qquad [D^\dagger_{ij}, D^\dagger_{kl}] = [D_{ij}, D_{kl}] = 0 \\ &[E_{ij}, D^\dagger_{kl}] = \delta_{jk}D^\dagger_{il} + \delta_{jl}D^\dagger_{ki} \qquad [E_{ij}, D_{kl}] = -\delta_{ik}D_{jl} - \delta_{il}D_{kj} \\ &[D_{ij}, D^\dagger_{kl}] = \delta_{ik}E_{lj} + \varepsilon\delta_{il}E_{kj} + \varepsilon\delta_{jk}E_{li} + \delta_{jl}E_{ki} \end{aligned} \tag{2.9}$$

and their symmetry and Hermiticity properties are given by

$$D^\dagger_{ij} = \varepsilon D^\dagger_{ji} \qquad D_{ij} = \varepsilon D_{ji} \tag{2.10}$$

and

$$(D^\dagger_{ij})^\dagger = D_{ij} \qquad (E_{ij})^\dagger = E_{ji}. \tag{2.11}$$

We can take as independent generators all the operators E_{ij}, as well as those operators $D^\dagger_{ij}$ and D_{ij} for which $i \leqslant j$ for $Sp(2d, R)$, or $i<j$ for $SO^*(2d)$. They separate into the following three subsets:

(i) E_{ii} $\quad i = 1, \ldots, d$ (2.12a)

(ii) E_{ij} $\quad i<j$

$D^\dagger_{ij}$ $\quad i \leqslant j$ for $Sp(2d, R)$ and $i<j$ for $SO^*(2d)$ (2.12b)

(iii) E_{ij} $\quad i>j$

D_{ij} $\quad i \leqslant j$ for $Sp(2d, R)$ and $i<j$ for $SO^*(2d)$ (2.12c)

respectively made of the weight, raising and lowering generators.

With the chains

$$U(n) \supset O(n) \qquad \text{and} \qquad U(n) \supset USp(n) \tag{2.13a, b}$$

† In fact, the $U(n)$ and $U(d)$ subgroups of $Sp(2dn, R)$ are respectively generated by the operators $E_{st} = \frac{1}{2}\sum_{i=1}^{d} (\eta_{is}\xi_{it} + \xi_{it}\eta_{is})$, and E_{ij}, which only differ from C_{st} and C_{ij} in some irrelevant constants, $\frac{1}{2}d\delta_{st}$ and $\frac{1}{2}n\delta_{ij}$.

we can therefore respectively associate the complementary chains

$$\mathrm{Sp}(2d, R) \supset \mathrm{U}(d) \qquad \text{and} \qquad \mathrm{SO}^*(2d) \supset \mathrm{U}(d). \tag{2.14a, b}$$

Within either irrep $\langle (1/2)^{dn} \rangle$ or $\langle (1/2)^{dn-1} 3/2 \rangle$ of $\mathrm{Sp}(2dn, R)$, the irreps of $\mathrm{U}(n)$ and $\mathrm{U}(d)$ are characterised by the same partition $[h_1 \ldots h_d]$. Those of $\mathrm{Sp}(2d, R)$ and $\mathrm{SO}^*(2d)$ are positive discrete series, specified by their lowest weight $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$, where $(\lambda_1 \ldots \lambda_d)$ is the partition characterising the irreps of $\mathrm{O}(n)$ and $\mathrm{USp}(n)$, respectively (it will be assumed that $d \leqslant \nu = [\frac{1}{2}n]$ throughout this paper). The lowest weight state $|\mathrm{LWS}\rangle$ of the $\mathrm{Sp}(2d, R)$ and $\mathrm{SO}^*(2d)$ irreps satisfies the following system of equations:

$$\begin{aligned} &D_{ij}|\mathrm{LWS}\rangle = 0 \\ &E_{ij}|\mathrm{LWS}\rangle = 0 \qquad i > j \\ &E_{ii}|\mathrm{LWS}\rangle = (\lambda_{d+1-i} + \tfrac{1}{2}n)|\mathrm{LWS}\rangle. \end{aligned} \tag{2.15}$$

In the reduction of the $\mathrm{Sp}(2dn, R)$ irreps, the $\mathrm{Sp}(2d, R)$ or $\mathrm{SO}^*(2d)$ irrep $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$ appears with a multiplicity equal to the dimension $\dim(\lambda)$ of the corresponding $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ irrep $(\lambda_1 \ldots \lambda_d)$. Equation (2.15) therefore has $\dim(\lambda)$ independent solutions, which can be characterised by their transformation properties under $\mathrm{O}(n)$ or $\mathrm{USp}(n)$, i.e. by a given row of the irrep $(\lambda_1 \ldots \lambda_d)$. All the $\mathrm{Sp}(2d, R)$ or $\mathrm{SO}^*(2d)$ bases obtained from one of these LWS correspond to the same row of $(\lambda_1 \ldots \lambda_d)$.

In the next section, we shall reformulate the state labelling problem for the chains (2.13a, b) in terms of the complementary chains (2.14a, b).

3. Reduction of the unitary group to its orthogonal or symplectic subgroup

When $d \leqslant \nu$, the decomposition of the d-row irrep $[h_1 h_2 \ldots h_d]$ of $\mathrm{U}(n)$ into irreps $(\lambda_1 \lambda_2 \ldots \lambda_d)$ of $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ is governed by Littlewood's branching rule (1950). It states that, if in the reduction to irreps of $\mathrm{U}(n)$, the product representation $[\lambda_1 \lambda_2 \ldots \lambda_d] \times [h_1^s h_2^s \ldots h_d^s]$ contains $[h_1 h_2 \ldots h_d]$ a certain number of times, which we denote by $g_{[\lambda][h^s][h]}$, then the irrep $[h_1 h_2 \ldots h_d]$ of $\mathrm{U}(n)$ breaks into irreps $(\lambda_1 \lambda_2 \ldots \lambda_d)$ of $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ according to the following relation:

$$[h_1 h_2 \ldots h_d] = \sum_{\lambda_1 \ldots \lambda_d} \left(\sum_{h_1^s \ldots h_d^s} g_{[\lambda][h^s][h]} \right) (\lambda_1 \lambda_2 \ldots \lambda_d) \tag{3.1}$$

where the summation over $h_1^s, \ldots, h_d^s$ is restricted to partitions into even parts in the $\mathrm{O}(n)$ case, and to partitions in which each part is repeated an even number of times in the $\mathrm{USp}(n)$ one. Practical evaluation of the branching rule is made easier by using infinite series of S functions (King 1975). When $d > \nu$, the non-standard $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ symbols $(\lambda_1 \ldots \lambda_d)$ have to be converted into standard ones by using Newell's modification rules (1951), thereby making the branching rule quite complicated. We shall therefore restrict ourselves here to the $d \leqslant \nu$ case, although the detailed study of the $d > \nu$ case, previously carried out for $\mathrm{O}(n)$ (Quesne 1984), could be extended to $\mathrm{USp}(n)$ along the lines of the present work.

Let us consider the highest weight states (HWS), $P(\eta_{is})|0\rangle$, of all the equivalent $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ irreps characterised by $(\lambda_1 \ldots \lambda_d)$, and contained in an irrep $[h_1 \ldots h_d]$ of $\mathrm{U}(n)$. Here $P(\eta_{is})$ is a polynomial in the dn boson creation operators η_{is}, and $|0\rangle$

denotes the boson vacuum state. Those HWS can be obtained as the simultaneous solutions of the following two systems of equations:

$$\Lambda_{s,-s}P(\eta_{is})|0\rangle = \lambda_{\nu+1-s}P(\eta_{is})|0\rangle \qquad s=\nu,\ldots,1 \tag{3.2a}$$

$$\Lambda_{st}P(\eta_{is})|0\rangle = 0 \qquad s>t>-s \text{ for } \mathrm{O}(n)$$
$$\qquad s\geqslant t>-s \text{ for } \mathrm{USp}(n) \tag{3.2b}$$

and

$$C_{ii}P(\eta_{is})|0\rangle = h_iP(\eta_{is})|0\rangle \qquad i=i,\ldots,d \tag{3.3a}$$

$$C_{ij}P(\eta_{is})|0\rangle = 0 \qquad i<j \tag{3.3b}$$

where we assume that $\lambda_{d+1},\ldots,\lambda_\nu$ are all equal to zero. Equations (3.2) and (3.3) can also be interpreted in terms of the reduction of the $\mathrm{Sp}(2d, R)$ or $\mathrm{SO}^*(2d)$ irreps with respect to $\mathrm{U}(d)$. To prove this assertion, let us successively analyse the significance of both these equations.

From equation (2.6), a solution of equation (3.2) is the HWS of some $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ irrep characterised by the partition $(\lambda_1\ldots\lambda_d)$. From the complementarity between $\mathrm{O}(n)$ and $\mathrm{Sp}(2d, R)$, or $\mathrm{USp}(n)$ and $\mathrm{SO}^*(2d)$, it follows that the set of all such solutions span the carrier space of an $\mathrm{Sp}(2d, R)$ or $\mathrm{SO}^*(2d)$ irrep specified by its lowest weight $\langle\lambda_d+\frac{1}{2}n,\ldots,\lambda_1+\frac{1}{2}n\rangle$. In the same way, a solution of equation (3.3) is the HWS of some $\mathrm{U}(d)$ irrep characterised by the partition $[h_1\ldots h_d]$. The complementarity between $\mathrm{U}(d)$ and $\mathrm{U}(n)$ then imposes that the set of all such solutions span the carrier space of a $\mathrm{U}(n)$ irrep specified by the same partition $[h_1\ldots h_d]$.

If we now consider all the simultaneous solutions of equations (3.2) and (3.3), we can interpret them either as the HWS of all the equivalent $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ irreps characterised by the same partition $(\lambda_1\ldots\lambda_d)$, and contained in a $\mathrm{U}(n)$ irrep $[h_1\ldots h_d]$, or as the HWS of all the equivalent $\mathrm{U}(d)$ irreps specified by the same partition $[h_1\ldots h_d]$, and contained in an $\mathrm{Sp}(2d, R)$ or $\mathrm{SO}^*(2d)$ irrep $\langle\lambda_d+\frac{1}{2}n,\ldots,\lambda_1+\frac{1}{2}n\rangle$. This establishes the equivalence of the state labelling problems for the chains (2.13a) and (2.14a), as well as (2.13b) and (2.14b).

Consequently, the simultaneous solutions of equations (3.2) and (3.3) can be written as the kets

$$P(\eta_{is})|0\rangle = \left|\begin{matrix}\langle\lambda_d+\frac{1}{2}n,\ldots,\lambda_1+\frac{1}{2}n\rangle \\ (\Gamma^s)[h_1\ldots h_d] \\ \max\end{matrix}\quad;\quad\begin{matrix}[h_1\ldots h_d] \\ (\Gamma^s)(\lambda_1\ldots\lambda_d) \\ \max\end{matrix}\right\rangle \tag{3.4}$$

whose left-hand part characterises the irreps of the chain (2.14a) or (2.14b), and whose right-hand part specifies those of the chain (2.13a) or (2.13b). Here (Γ^s) denotes the whole set of k missing labels, distinguishing between repeated irreps of $\mathrm{U}(d)$ contained in a given irrep of $\mathrm{Sp}(2d, R)$ or $\mathrm{SO}^*(2d)$, as well as between repeated irreps of $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ contained in a given irrep of $\mathrm{U}(n)$. The number k of missing labels is respectively given by

$$k=\tfrac{1}{2}d(d-1) \tag{3.5a}$$

for the chains (2.13a) and (2.14a), and

$$\begin{aligned} k&=\tfrac{1}{2}d(d-3) && \text{if } d\geqslant 3 \\ &=0 && \text{if } d=1,2 \end{aligned} \tag{3.5b}$$

for the chains (2.13b) and (2.14b).

Equations (3.5a) and (3.5b) can be established for the chains (2.13a) and (2.13b) by applying a result of Seligman and Sharp (1983). The latter states that the number of internal labels needed to specify the states of a degenerate irrep (i.e. an irrep for which one or more labels are zero) of a compact semi-simple Lie group is given by

$$b=\tfrac{1}{2}(r-l)-a \tag{3.6}$$

where r and l are the order and the rank of the group, respectively, and a is the number of lowering generators which annihilate the irrep HWS. For d-row irreps of the group $\mathrm{G}=\mathrm{U}(n)$, we easily find from the well known expression of their HWS (Moshinsky 1963) that

$$b_{\mathrm{G}}=\tfrac{1}{2}d(2n-d-1) \tag{3.7}$$

while for d-row irreps of the subgroup $\mathrm{H}=\mathrm{O}(n)$ or $\mathrm{USp}(n)$, we find from equations (2.6c) and (5.2) below that

$$\begin{aligned} b_{\mathrm{H}} &= d(n-d-1) \qquad && \text{for } \mathrm{O}(n)\\ &= d(n-d) && \text{for } \mathrm{USp}(n). \end{aligned} \tag{3.8}$$

The number k of missing labels is then given by

$$k=b_{\mathrm{G}}-b_{\mathrm{H}}-\bar{d} \tag{3.9}$$

where $\bar{d}$ is the number of independent labels characterising the subgroup irrep $(\lambda_1\ldots\lambda_d)$, and is equal to d except for $\mathrm{USp}(n)$ and $d=1,2$, for which, as a consequence of Littlewood's branching rule (3.1), it reduces to 0 and 1, respectively. Equations (3.7), (3.8) and (3.9) finally lead to equation (3.5).

In § 5, we shall show how a canonical choice can be made for the k missing labels (Γ^s). For such a purpose, we shall construct the whole set of simultaneous solutions of equations (3.2) and (3.3). As it will prove convenient to know the latter for the special case where the irrep of $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ is the scalar one, we first study this case in detail in the next section.

4. The case of scalar representations of the orthogonal or symplectic subgroup

From Littlewood's branching rule (3.1), it results that the scalar irrep (0) of $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ is contained with multiplicity one in the $\mathrm{U}(n)$ irreps $[h_1\ldots h_d]$ for which $h_1,\ldots,h_d$ are even integers in the $\mathrm{O}(n)$ case, or $h_{2i}=h_{2i-1}$, $i=1,\ldots,\delta=[\tfrac{1}{2}d]$, and $h_d=0$ if d is odd, in the $\mathrm{USp}(n)$ one, and that it does not appear in the remaining irreps. We shall henceforth denote the $\mathrm{U}(n)$ irreps containing the scalar irrep of $\mathrm{O}(n)$ or $\mathrm{USp}(n)$ by the symbol $[h_1^s\ldots h_d^s]$.

Let us consider the system of equations (3.2) and (3.3), where $\lambda_{\nu+1-s}$ and h_i are replaced by 0 and h_i^s, respectively, and search for its single simultaneous solution. From § 3, the latter is the HWS of a $\mathrm{U}(d)$ irrep $[h_1^s\ldots h_d^s]$, belonging to the carrier space of the single $\mathrm{Sp}(2d,R)$ or $\mathrm{SO}^*(2d)$ irrep $\langle(\tfrac{1}{2}n)^d\rangle$ appearing in the reduction of the $\mathrm{Sp}(2dn,R)$ irreps. The LWS of the irrep $\langle\tfrac{1}{2}n)^d\rangle$, i.e. the solution of equation (2.15) where λ_{d+1-i} is replaced by 0, is the boson vacuum state $|0\rangle$. The remaining bases are generated from it by applying polynomials in the $D_{ij}^{\dagger}$, E_{ij} and D_{ij} generators. By using the commutation relations (2.9), it is always possible to write such polynomials in the normal form, i.e. as

$$P(D^\dagger_{ij})P'(E_{ij})P''(D_{ij}) \tag{4.1}$$

where P, P' and P'' are some polynomials in the indicated operators. Since the operators D_{ij} annihilate the state $|0\rangle$, while the operators E_{ij} reduce to the constants $\frac{1}{2}n\ \delta_{ij}$ when acting upon the latter, the bases of $\langle(\frac{1}{2}n)^d\rangle$ can be written as $P(D^\dagger_{ij})|0\rangle$.

It remains to determine the polynomial $P^s(D^+_{ij})$, which creates the HWS of the $U(d)$ irrep $[h^s_1, \ldots, h^s_d]$,

$$P^s(D^\dagger_{ij})|0\rangle = \left|\begin{matrix}\langle(\frac{1}{2}n)^d\rangle & [h^s_1 \ldots h^s_d] \\ [h^s_1 \ldots h^s_d]; & (0) \\ \text{max} & \end{matrix}\right\rangle. \tag{4.2}$$

In equation (4.2), no additional labels (Γ^s) are needed. The explicit form of $P^s(D^\dagger_{ij})$ can be found by solving equation (3.3), where h_i is replaced by h^s_i, and $P(\eta_{is})$ by $P^s(D^\dagger_{ij})$. This is most easily done by applying the method of elementary permissible diagrams (EPD) (Moshinsky and Syamala Devi 1969, Sharp and Lam 1969).

In the $O(n)$ case, there are d EPD, corresponding to the $U(n)$ irreps $[2^i]$, $i = 1, \ldots, d$, respectively. Their HWS can be written as

$$D^\dagger_{12\ldots i,12\ldots i} = \sum_p (-1)^p D^\dagger_{1,p(1)} D^\dagger_{2,p(2)} \ldots D^\dagger_{i,p(i)} \tag{4.3}$$

where the summation is carried out over the $i!$ permutations of the indices $1, 2, \ldots, i$. In terms of them, the polynomial $P^s(D^+_{ij})$ reads (Deenen and Quesne 1982)

$$P^s(D^\dagger_{ij}) = \prod_{i=1}^{d} (D^\dagger_{12\ldots i,12\ldots i})^{(h_i - h_{i+1})/2} \tag{4.4}$$

where h_{d+1} is assumed to be equal to zero.

In the $USp(n)$ case, there are δ EPD, corresponding to the $U(n)$ irreps $[1^{2i}]$, $i = 1, \ldots, \delta$, respectively. Their HWS are given by

$$D^\dagger_{1\,2\ldots 2i-1\,2i} = \sum_p (-1)^p D^\dagger_{p(1),p(2)} \ldots D^\dagger_{p(2i-1),p(2i)} \tag{4.5}$$

where the summation is carried out over the $[2^i i!]^{-1}(2i)! = (2i-1)!!$ permutations of the indices $1, \ldots, 2i$, which exchange neither the indices of the same $D^\dagger$ operator nor the pairs of indices of two such operators. For $i = 4$, for instance, equation (4.5) reads

$$D^\dagger_{1234} = D^\dagger_{12}D^\dagger_{34} - D^\dagger_{13}D^\dagger_{24} + D^\dagger_{14}D^\dagger_{23}. \tag{4.6}$$

The polynomial $P^s(D^\dagger_{ij})$ can then be written as

$$P^s(D^\dagger_{ij}) = \prod_{i=1}^{\delta} (D^\dagger_{1\,2\ldots 2i-1\,2i})^{h_{2i-1} - h_{2i+1}} \tag{4.7}$$

where h_{d+1} is again assumed to be equal to zero.

Having solved equations (3.2) and (3.3) for the special case of $O(n)$ or $USp(n)$ scalar irreps, we shall now proceed to the general case in the next section.

5. The general case

Let us consider equations (3.2) and (3.3), where $h_1, \ldots, h_d$, and $\lambda_1, \ldots, \lambda_d$ now assume arbitrary values compatible with Littlewood's branching rule (3.1). From § 3, their simultaneous solutions are the HWS of the equivalent U(d) irreps, characterised by the same partition $[h_1 \ldots h_d]$, and belonging to a given Sp($2d$, R) or SO*($2d$) irrep $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$, namely that made of O($n$) or USp($n$) HWS. Let us therefore first construct the carrier space of the latter, and then search for the U(d) HWS it contains.

The LWS of the Sp($2d$, R) or SO*($2d$) irrep $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$, made of O($n$) or USp($n$) HWS, is the simultaneous solution of equations (2.15) and (3.2). In the notations of equation (3.4), it can be written as

$$\bar{P}(\eta_{is})|0\rangle = \left| \begin{matrix} \langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle & & [\lambda_1 \ldots \lambda_d] \\ [\lambda_1 \ldots \lambda_d] & ; & (\lambda_1 \ldots \lambda_d) \\ \min & & \max \end{matrix} \right\rangle \tag{5.1}$$

where no additional labels (Γ^s) are needed. The explicit form of $\bar{P}(\eta_{is})$ is easily found to be

$$\bar{P}(\eta_{is}) = \prod_{i=1}^{d} (\eta_{d-i+1 \ldots d, -\nu+i-1 \ldots -\nu})^{\lambda_i - \lambda_{i+1}}. \tag{5.2}$$

In equation (5.2), $\eta_{d-i+1 \ldots d, -\nu+i-1 \ldots -\nu}$ is defined by

$$\eta_{d-i+1 \ldots d, -\nu+i-1 \ldots -\nu} = \sum_{p} (-1)^p \eta_{d-i+1, p(-\nu+i-1)} \cdots \eta_{d, p(-\nu)} \tag{5.3}$$

where the summation is carried out over the $i!$ permutations of the indices $-\nu+i-1, \ldots, -\nu$.

From the LWS (5.1), we can generate all the bases of the irrep $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$ by applying polynomials of the form given in equation (4.1). In this equation, we can eliminate $P''(D_{ij})$ since the operators D_{ij} annihilate the LWS. Moreover, the action of all the polynomials $P'(E_{ij})$ upon the latter generates the carrier space of the U(d) irrep $[\lambda_1 \ldots \lambda_d]$. It is possible to choose $P'(E_{ij})$ in such a way that the resulting state transforms irreducibly under the canonical chain $U(d) \supset U(d-1) \supset \ldots \supset U(1)$, and is characterised by a Gel'fand pattern (λ) (Gel'fand and Tseitlin 1950, Baird and Biedenharn 1963, Moshinsky 1963). In the notation of equation (3.4), such a state can be written as

$$\bar{P}_{(\lambda)}(\eta_{is})|0\rangle = \left| \begin{matrix} \langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle & & [\lambda_1 \ldots \lambda_d] \\ [\lambda_1 \ldots \lambda_d] & ; & (\lambda_1 \ldots \lambda_d) \\ (\lambda) & & \max \end{matrix} \right\rangle . \tag{5.4}$$

In practice the explicit form of $\bar{P}_{(\lambda)}(\eta_{is})$ can be found by applying appropriate U(d), U($d-1$), ..., U(2) raising operators (Nagel and Moshinsky 1965) to the state (5.1). Finally, all the bases of the irrep $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$ are obtained from the set of states (5.4) by applying all possible polynomials $P(D_{ij}^{\dagger})$.

It remains to find the HWS of all the equivalent U(d) irreps, characterised by the same partition $[h_1 \ldots h_d]$, in the carrier space of the irrep $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$. They are obtained by solving equation (3.3), where $P(\eta_{is})$ is a linear combination of the $\langle \lambda_d + \frac{1}{2}n, \ldots, \lambda_1 + \frac{1}{2}n \rangle$ bases we have just constructed. Since the states (5.4) already transform irreducibly under U(d), let us classify the polynomials $P(D_{ij}^{\dagger})$ according to U(d) irreps. From § 4, we know that, when applied to the boson vacuum state, the set of all polynomials $P(D^{\dagger}_{ij})$ gives rise to the bases of the $\mathrm{Sp}(2d, R)$ or $\mathrm{SO}^*(2d)$ irrep $\langle(\frac{1}{2}n)^d\rangle$. A set of linearly independent polynomials in $D^{\dagger}_{ij}$ can therefore be obtained by considering all allowed values of $h^s_1, \ldots, h^s_d$, as specified in § 4, and all Gel'fand patterns (h^s) corresponding to $[h^s_1 \ldots h^s_d]$. In the notations of equation (3.4), the corresponding states can be written as

$$P^s_{(h^s)}(D^{\dagger}_{ij})|0\rangle = \left| \begin{matrix} \langle(\frac{1}{2}n)^d\rangle & [h^s_1 \ldots h^s_d] \\ [h^s_1 \ldots h^s_d]\ ; & (0) \\ (h^s) & \end{matrix} \right\rangle. \tag{5.5}$$

The explicit form of $P^s_{(h^s)}(D^{\dagger}_{ij})$ can be found by applying appropriate lowering operators (Nagel and Moshinsky 1965) to the HWS, given in equations (4.4) and (4.7).

It is now straightforward to obtain all the solutions (3.4) of equation (3.3). For such a purpose, we just have to couple the polynomials $\bar{P}_{(\lambda)}(\eta_{is})$ and $P^s_{(h^s)}(D^{\dagger}_{ij})$ to a definite irrep $[h_1 \ldots h_d]$ of $\mathrm{U}(d)$ by means of appropriate $\mathrm{U}(d)$ Wigner coefficients, as follows:

$$P(\eta_{is})|0\rangle = \sum_{(h^s)(\lambda)} \left\langle \begin{matrix} [h_1 \ldots h_d] \\ \max \end{matrix} \middle| \left\langle \begin{matrix} (\gamma^s) \\ [h^s_1 \ldots h^s_d] \\ (h^s) \end{matrix} \right\rangle \middle| \begin{matrix} [\lambda_1 \ldots \lambda_d] \\ (\lambda) \end{matrix} \right\rangle P^s_{(h^s)}(D^{\dagger}_{ij}) \bar{P}_{(\lambda)}(\eta_{is})|0\rangle. \tag{5.6}$$

In equation (5.6), we use Biedenharn's canonical characterisation of the $\mathrm{U}(d)$ Wigner coefficients by means of operator patterns (γ^s) (Biedenharn *et al* 1967). By this procedure, we identify the set of missing labels (Γ^s) with the irrep labels $h^s_1, \ldots, h^s_d$, and the operator patterns (γ^s), i.e.

$$(\Gamma^s) = \begin{pmatrix} (\gamma^s) \\ [h^s_1 \ldots h^s_d] \end{pmatrix}. \tag{5.7}$$

Since the partitions $[h^s_1 \ldots h^s_d]$ are those appearing in Littlewood's branching rule (3.1), and moreover (γ^s) solves the state labelling problem for the product $[\lambda_1 \ldots \lambda_d] \times [h^s_1 \ldots h^s_d]$ of $\mathrm{U}(d)$ irreps, the number of the states (5.6), corresponding to all possible (Γ^s), agrees with that predicted by Littlewood's branching rule. We have therefore found all the simultaneous solutions of equations (3.2) and (3.3) by reducing the internal state labelling problem for $\mathrm{U}(n) \supset \mathrm{O}(n)$ or $\mathrm{USp}(n)$ to the external state labelling problem for $\mathrm{U}(d)$.

As a final check, let us show that the definition (5.7) of (Γ^s) provides us with the right number k of additional labels, as given in equation (3.5). In equation (5.7), the total number of labels Γ^s_{ji}, $i \leq j \leq i \leq d$, is equal to $\frac{1}{2}d(d+1)$; however, they are linked by the d relations

$$\sum_{j=1}^{i} \Gamma^s_{ji} - \sum_{j=1}^{i-1} \Gamma^s_{ji-1} = h_i - \lambda_i \qquad i = 1, \ldots, d. \tag{5.8}$$

In the $\mathrm{O}(n)$ case, there are no other relations among the Γ^s_{ji}, so that the number of independent labels is given by $\frac{1}{2}d(d-1)$, as predicted by equation (3.5*a*). In the $\mathrm{USp}(n)$ case, however, for $d \geq 3$ there are d additional relations among the Γ^s_{ji}, namely the conditions

$$\begin{aligned} &h^s_{2i} = h^s_{2i-1} && i = 1, \ldots, \delta \\ &h^s_d = 0 && \text{if } d \text{ is odd} \end{aligned} \tag{5.9}$$

resulting from the definition of $[h_1^s \ldots h_d^s]$, and the relations

$$\gamma_{2i-1,d-1}^s = h_{2i-1}^s \qquad i = 1, \ldots, \delta \tag{5.10}$$

following from the triangular inequalities satisfied by operator patterns (Biedenharn *et al* 1967); the number of independent labels Γ_{ji}^s is therefore given by $\frac{1}{2}d(d-3)$, in agreement with equation (3.5*b*). For $d = 1$ or 2, a similar count also leads to equation (3.5*b*).

In conclusion, we have proved that the previously proposed canonical solution of the state labelling problem for $U(n) \supset O(n)$ (Deenen and Quesne 1983, Quesne 1984) can indeed be extended to $U(n) \supset USp(n)$, provided we switch from the metric $g = \boldsymbol{I}$, specific to $O(n)$, to the unified metric (2.1), and replace the complementary chain $Sp(2d, R) \supset U(d)$ by the corresponding chain $SO^*(2d) \supset U(d)$.

References

Baird G E and Biedenharn L C 1963 *J. Math. Phys.* **4** 1449
Biedenharn L C, Giovannini A and Louck J D 1967 *J. Math. Phys.* **8** 691
Deenen J and Quesne C 1982 *J. Math. Phys.* **23** 2004
—— 1983 *J. Phys. A: Math. Gen.* **16** 2095
Gelbart S 1979 *Proc. Symp. Pure Mathematics (Am. Math. Soc.)* **33** 287
Gel'fand I M and Tseitlin M L 1950 *Dokl. Akad. Nauk SSR* **71** 825
Gross K and Kunze R 1977 *J. Funct. Anal.* **25** 1
Howe R 1979 *Proc. Symp. Pure Mathematics (Am. Math. Soc.)* **33** 275
Kashiwara M and Vergne M 1978 *Invent. Math.* **44** 1
King R C 1975 *J. Phys. A: Math. Gen.* **8** 429
Littlewood D E 1950 *The Theory of Group Characters* (Oxford: Clarendon)
Moshinsky M 1963 *J. Math. Phys.* **4** 1128
Moshinsky M and Quesne C 1970 *J. Math. Phys.* **11** 1631
—— 1971 *J. Math. Phys.* **12** 1772
Moshinsky M and Syamala Devi V 1969 *J. Math. Phys.* **10** 455
Newell M J 1951 *Proc. R. Irish Acad.* **54A** 143, 153
Quesne C 1984 *J. Phys. A: Math. Gen.* **17** 777, 791
—— 1985 *J. Phys. A: Math. Gen.* **18** 1167
Seligman T H and Sharp R T 1983 *J. Math. Phys.* **24** 769
Sharp R T and Lam C S 1969 *J. Math. Phys.* **10** 2033